Difference of Analytic Sets

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Pointclasses

Definition: A class of subsets of a topological space is called a **pointclass** iff it is closed under preimages by continuous functions.

Notation: For Γ a pointclass of a space X, we set:

$$\check{\Gamma} = \left\{ A \subseteq X \mid A^{\complement} \in \Gamma \right\}$$

and:

$$\Delta(\Gamma) = \Gamma \cap \check{\Gamma}$$

We will say that Γ is **self-dual** if $\Gamma = \check{\Gamma}$.

Pointclasses

Examples:

$$
\begin{array}{ccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
\Delta^0_1 & & & & \Delta^0_2 & & & \cdots \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
 & & \Pi^0_1 & & & & \Pi^0_2
\end{array}
$$

René Baire

Pointclasses

Examples:

$$\begin{array}{ccccccccccc}
\Sigma^0_1 & & & & D_2(\Sigma^0_1) & & & & D_3(\Sigma^0_1) & & \\
& \subseteq & & \supseteq & & \subseteq & & \supseteq & & & \\
& & \Delta(D_2(\Sigma^0_1)) & & & & \Delta(D_3(\Sigma^0_1)) & & & & \cdots \\
& \supseteq & & \subseteq & & \supseteq & & \subseteq & & & \\
\Pi^0_1 & & & & \check{D}_2(\Sigma^0_1) & & & & \check{D}_3(\Sigma^0_1) & &
\end{array}$$

Pointclasses

Examples:

Σ^1_1 Σ^1_2

Δ^1_1 Δ^1_2 $\cdots$

Π^1_1 Π^1_2

Reduction by continuous functions

Let A and B be two subsets of a topological space X. We say that A is (Wadge) reducible to B iff there exists a continuous function $f : X \to X$ such that:

$$f^{-1}(B) = A$$

We denote it by:

$$A \leq_W B$$

Reduction by continuous functions

Complexity seen as the *membership problem*:

Reduction by continuous functions

Complexity seen as the *membership problem*:

If $A \leq_W B$, the question:

$$x \in A \ ?$$

Becomes:

$$f(x) \in B \ ?$$

Reduction by continuous functions

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$$\Downarrow$$

$\leq_W$ is **reflexive**.

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Composition of two continuous functions is continuous

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$\leq_W$ is **transitive**.

Reduction by continuous functions

$\leq_W$ is **reflexive**

&

$\leq_W$ is **transitive**

$\Longrightarrow$ $\leq_W$ is a **preorder**!

Wadge Degrees & Wadge Hierarchy

The preorder $\leq_W$ induces an equivalence relation on $\mathcal{P}(X)$ whose equivalence classes are called **Wadge degrees**.

The collection of Wadge degrees together with the induced order is called the **Wadge hierarchy**.

Wadge Game

By now we will restrict ourselves to the Baire space ω^ω, with the usual topology.

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In this framework we have a very useful tool: the **Wadge Game**.

Wadge Games

Wadge Games

- 2 players infinite games;

Wadge Games

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- With perfect information;

Wadge Games

- 2 players infinite games;
- With perfect information;
- No chance.

Wadge Games

$W(A, B)$:

I : x_0

II :

Wadge Games

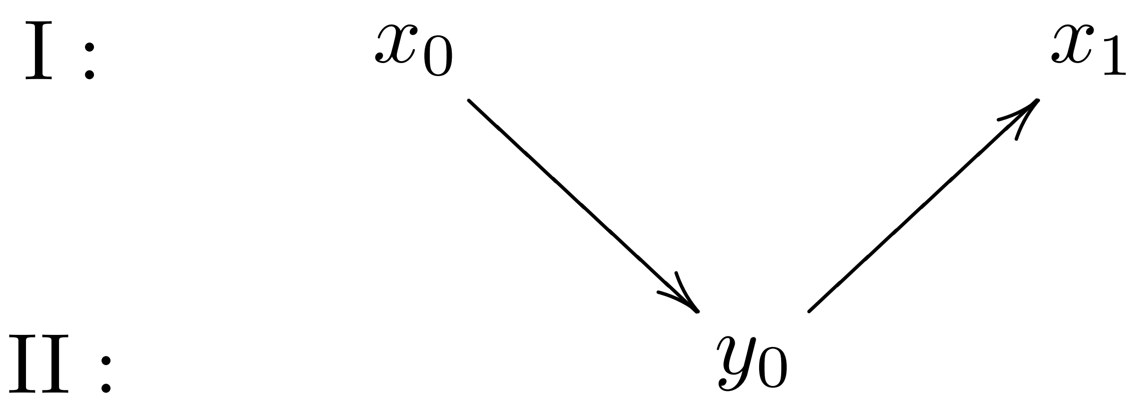
$W(A, B)$:

I : x_0

II : y_0

Wadge Games

$W(A, B)$:



Wadge Games

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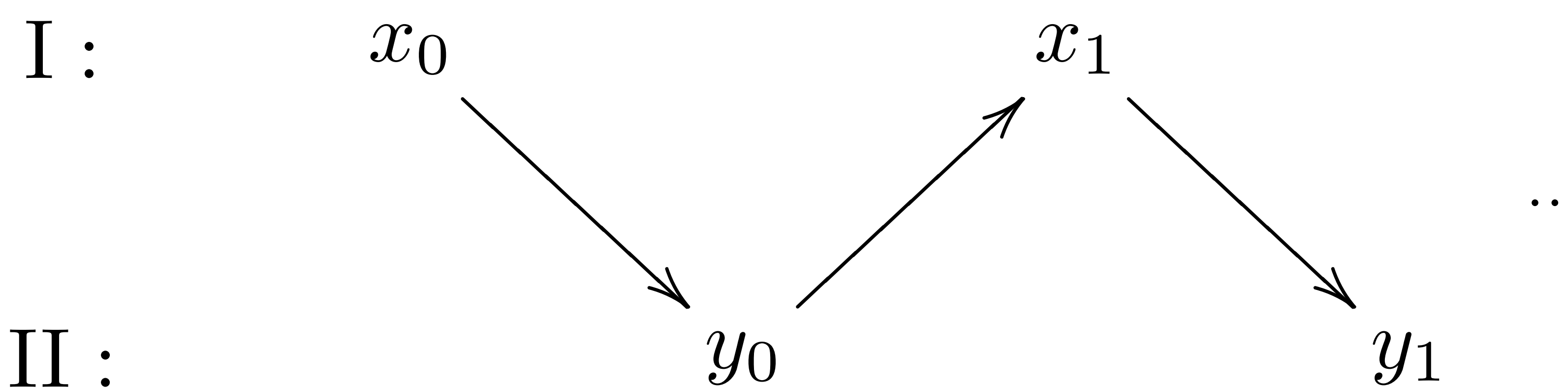

Wadge Games

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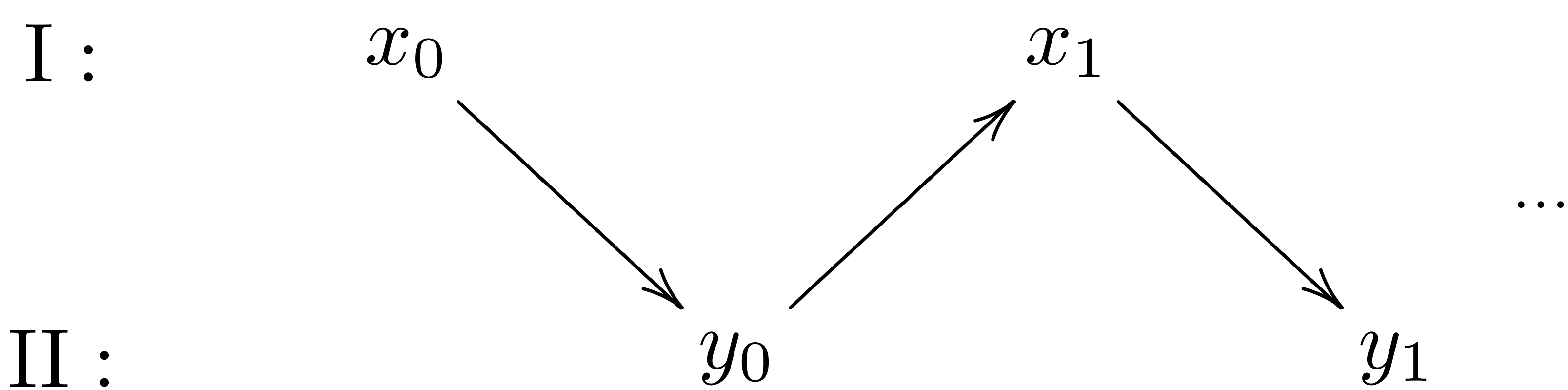

II can skip and wins iff $x \in A \leftrightarrow y \in B$

Continuous functions of the Baire Space

An application $f : \omega^\omega \longrightarrow \omega^\omega$ is continuous iff it arises from

$$\varphi : \omega^{<\omega} \longrightarrow \omega^{<\omega}$$

That is **monotone :**

$$s \subseteq t \Longrightarrow \varphi(s) \subseteq \varphi(t)$$

Continuous functions of the Baire Space

An application $f : \omega^\omega \longrightarrow \omega^\omega$ is continuous iff it arises from

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That is monotone and **proper**:

$$\forall x \in \omega^\omega, \lim_{x\to\infty} \varphi(x \upharpoonright n) \in \omega^\omega$$

Wadge Games

II has a winning strategy in $W(A, B)$

$$\iff$$

$$A \leq_W B$$

Antichains?

Do we have $\emptyset \leq_W \omega^\omega$? NO!

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We have antichains! How bad is it?

Wadge's Lemma

If the game $W(A, B)$ is determined, then we have either:

$$B^{\mathsf{c}} \leq_W A \qquad \text{OR} \qquad A \leq_W B$$

(This result is also called the **Semi Linear Order** principle.)

Antichains

If A and B are incompatible, we have:

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But $A^{\mathsf{C}} \leq_W B$ is equivalent to $A \leq_W B^{\mathsf{C}}$!

Thus $A \leq_W B^{\mathsf{C}} \leq_W A$, so that $A \equiv_W B^{\mathsf{C}}$.

Antichains

Wadge's Lemma implies thus that if you restrict yourself to a determined pointclass, every self-dual degree is comparable to any other degree, and that if two degrees are incomparable, they must be dual to each other.

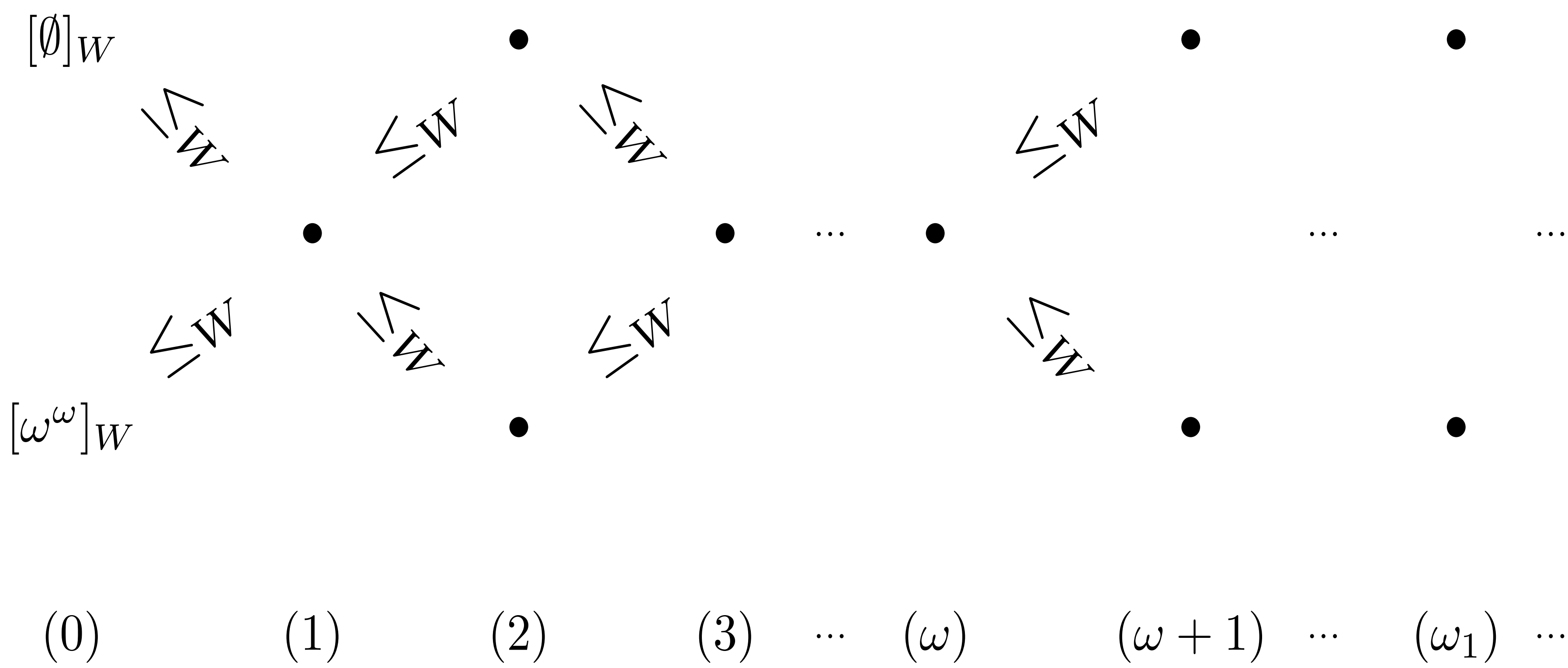
In that case, antichains have size at most 2!

Wellfoundedness

(Martin-Monk Theorem)

If we restrict ourselves to a pointclass Γ with appropriate closure and determinacy properties, then $\leq_W$ is well-founded.

Wadge Hierarchy



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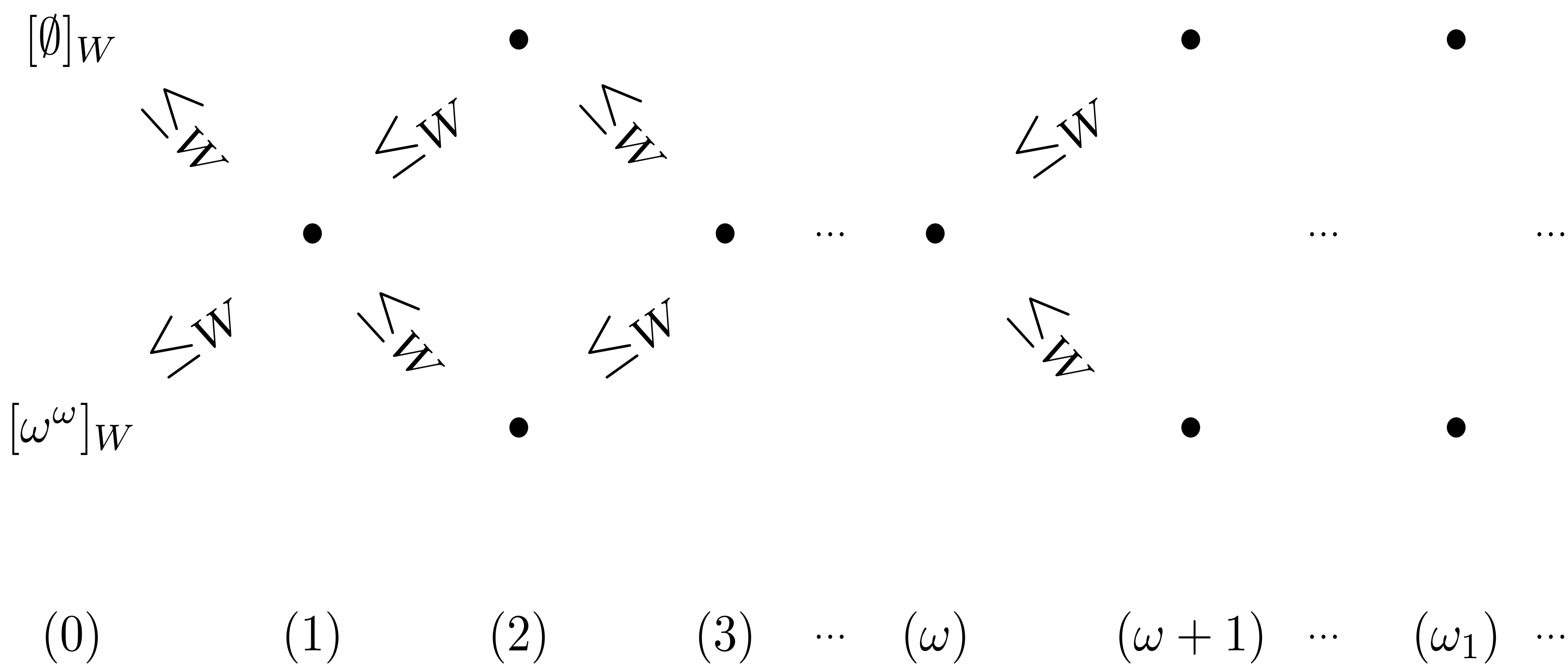

In ZFC, you can prove that this picture is accurate for the Borel sets!

Reduction by continuous functions

Notice that a class Γ is a pointclass iff it is an initial segment for $\leq_W$.

If Γ is a Borel pointclass, then it is either of the form:

$$\Gamma = \{B \subseteq X \mid B <_W A\}$$

or:

$$\Gamma = \{B \subseteq X \mid B \leq_W A\}$$

with $A \subseteq X$.

Application to properness

If you want to prove that a set $A \in \Sigma^0_\alpha$ is *proper* for that class, then you only have to prove that for any set $B \in \Sigma^0_\alpha$, II has a winning strategy in $W(B,A)$.

E.g.:

- prove that S_1 is Σ^0_1 proper:

$$S_1 := \{x \in \omega^\omega \mid \exists n \in \omega, x(n) = 0\}$$

- prove that P_2 is Π^0_2 proper:

$$P_2 := \{x \in \omega^\omega \mid \forall n\ \exists m \geq n, x(m) = 0\}$$

Wadge Hierarchy of Borel Sets

Definition: A **Boolean operation** is an application

$$\mathcal{O} : \mathcal{P}(\omega^\omega)^\omega \to \mathcal{P}(\omega^\omega)$$

assigning a new set to a countable sequence of sets, and with the property that there is a $T_{\mathcal{O}} \subseteq \mathcal{P}(\omega)$ such that for any $(A_n)_{n\in\omega}$:

$$\forall x \in \omega^\omega \, (x \in \mathcal{O}((A_n)_{n\in\omega}) \longleftrightarrow \{n \in \omega \mid x \in A_n\} \in T_{\mathcal{O}})$$

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Example: The countable union will be given by the truth table:

$$T_{\bigcup_\omega} = \mathcal{P}(\omega)\backslash\emptyset$$

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Example: The complement will be given by the truth table:

$$T_{\complement} = \mathcal{P}(\omega \backslash \{0\})$$

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$$\forall x \in \omega^\omega \, (x \in \mathcal{O}((A_n)_{n\in\omega}) \longleftrightarrow \{n \in \omega \mid x \in A_n\} \in T_\mathcal{O})$$

Theorem (Wadge): Each non-self-dual Borel pointclass in ω^ω is of the form

$$\{\mathcal{O}((A_n)_{n\in\omega}) \mid \forall n (A_n \in \Sigma^0_1)\}$$

with $T_\mathcal{O}$ Borel.

Wadge Hierarchy of Borel Sets

SOME RESULTS IN THE WADGE HIERARCHY OF BOREL SETS

A. Louveau

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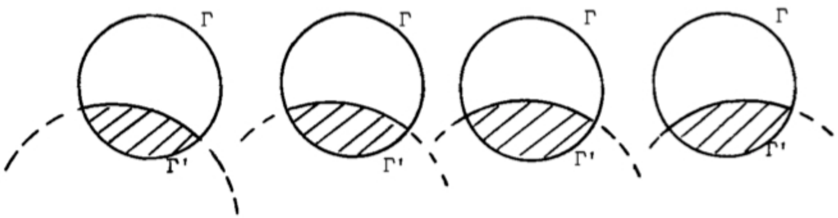
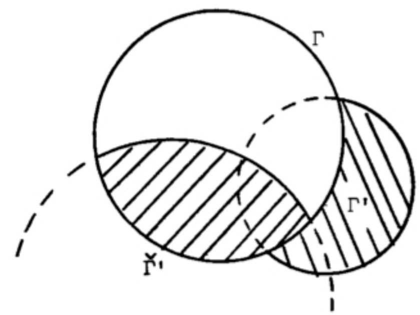


b. SU(Γ, Γ')



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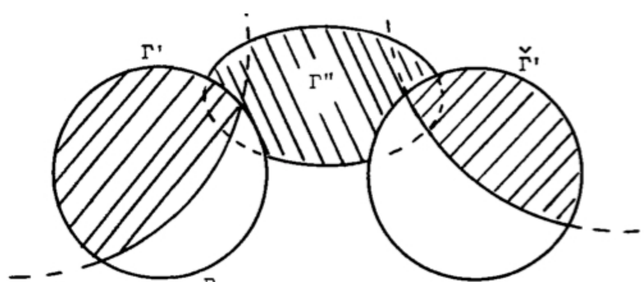

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Wadge Hierarchy of Borel Sets

WADGE HIERARCHY AND VEBLEN HIERARCHY

J. DUPARC

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$$B + A = A \cup \{u^\frown\langle a\rangle^\frown\beta :\ u \in \Lambda_A^{<\omega},\quad (a \in \Lambda_+ \textit{ and } \beta \in B) \textit{ or } (a \in \Lambda_- \textit{ and } \beta \notin B)\}$$

Let $A \subseteq \Lambda_A^\omega$, $B \subseteq \Lambda_B^\omega$, *both Borel and non self dual,* $d_w^\circ(A + B) = (d_w^\circ A) + (d_w^\circ B)$

Let $A \subseteq \Lambda_A^{\leq\omega}$,

- $A \bullet 1 = A$
- $A \bullet (v + 1) = (A \bullet v) + A$
- $A \bullet \lambda = \sup_{\theta \in \lambda} A \bullet \theta$, *for* λ *limit.*

Defines operations <u>on</u> sets which yield complete sets for all non-self-dual Borel pointclasses.

Generalizations?

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- Beyond Borel subsets!

Analytic Sets

- Analytic sets are the projections of Borel sets their complements are calles coanalytic sets;
- It is a pointclass closed under countable union and countable intersection.
- The sets that are analytic and co-analytic are called bi-analytic.

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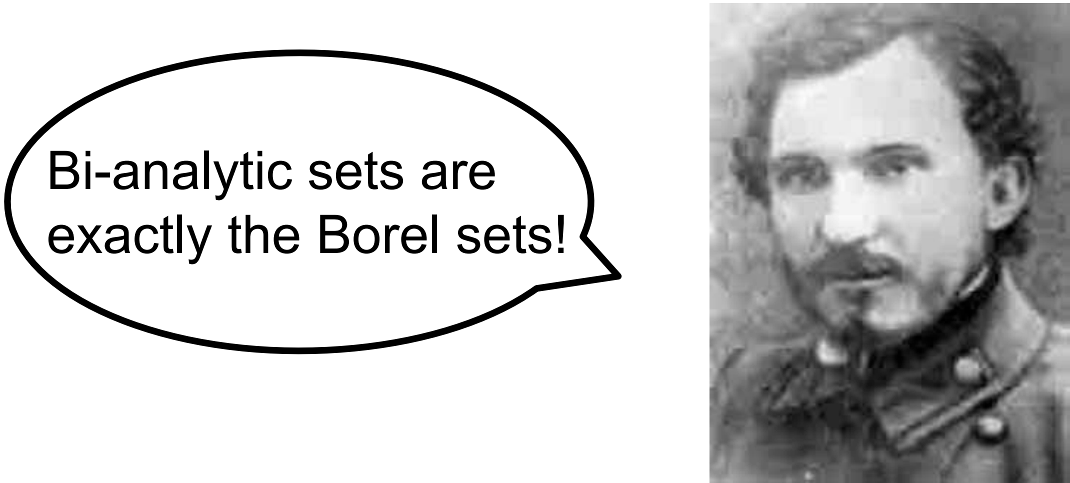

Structural Properties

Definition:

Let Γ be a class of sets. We say that Γ has the **separation property** if for any $A, B \in \Gamma$ with $A \cap B = \emptyset$, there is $C \in \Delta(\Gamma)$ separating A from B.

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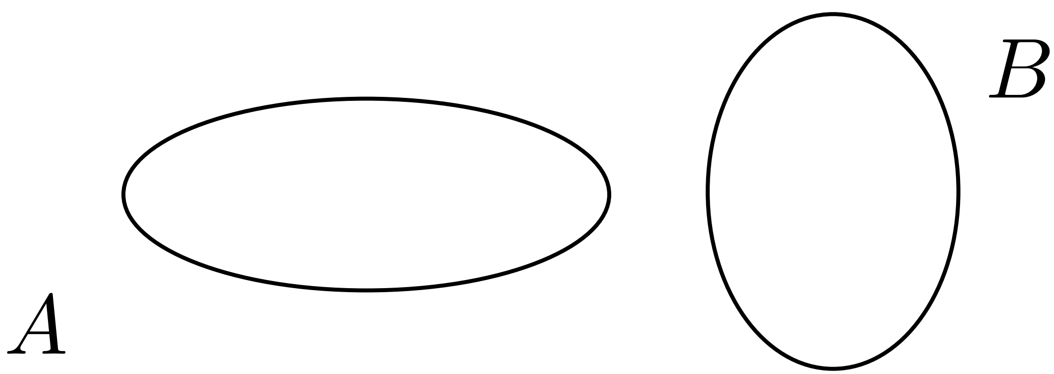

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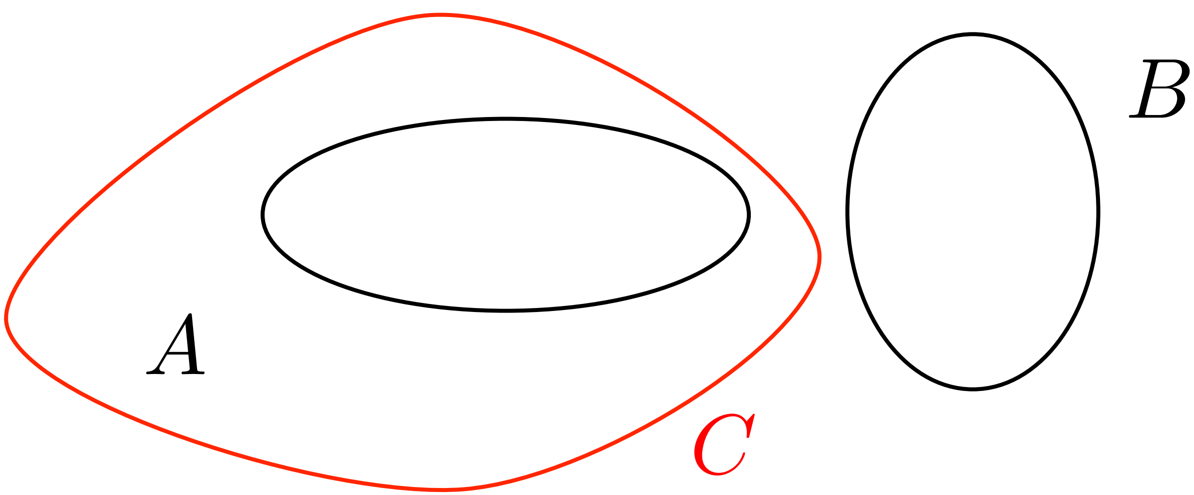

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<u>Definition</u>:

Let Γ be a class of sets. We say that Γ has the **generalized reduction property** if for any sequence $A_n \in \Gamma$, there is a *disjoint* sequence $A_n' \in \Gamma$ such that for all n, $A_n' \subseteq A_n$ and

$$\bigcup_{n\in\omega} A_n = \bigcup_{n\in\omega} A_n'.$$

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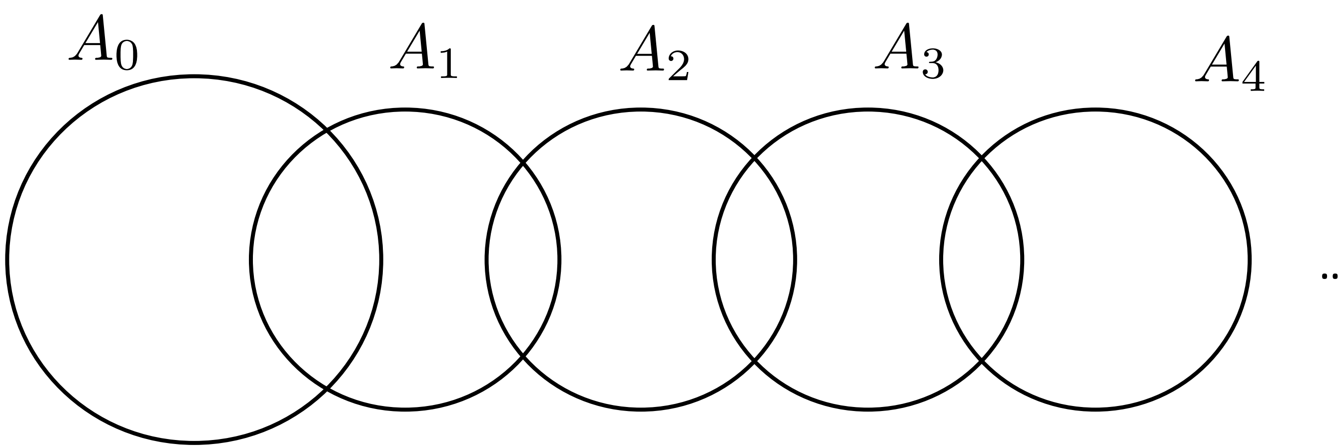

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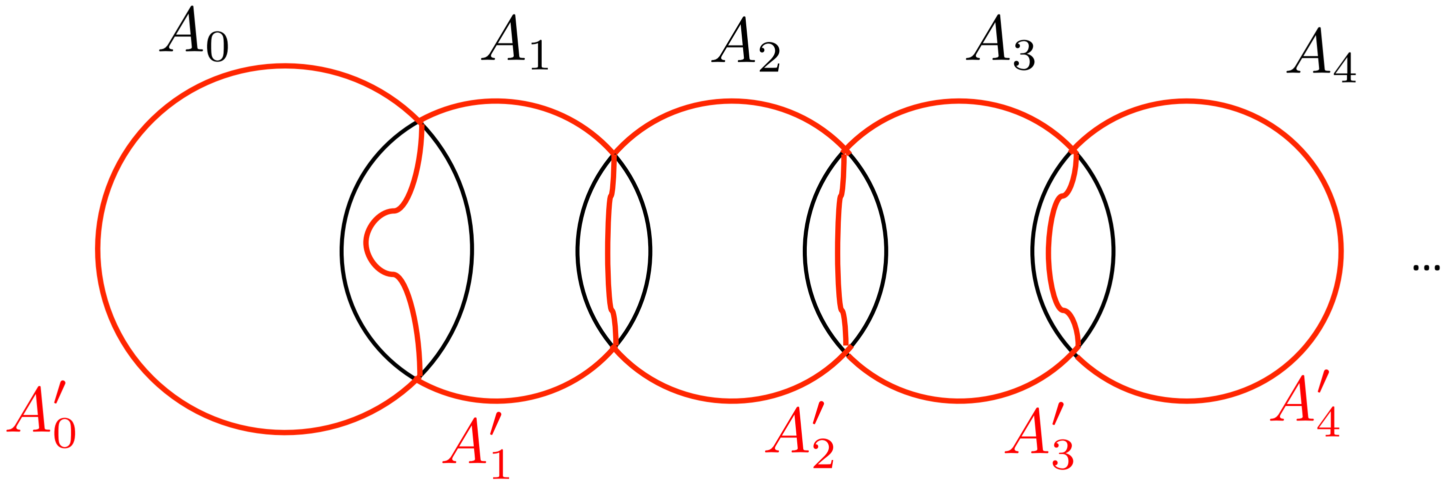

Structural Properties

Propositions:

The class of the analytic subsets of the Baire space has the **separation property**.

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The classe of the co-analytic subsets of the Baire space has **the generalized reduction property**.

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Fact:

Assuming Σ^1_1- determinacy, if $A \in \Sigma^1_1 \setminus \Pi^1_1$ then A is Σ^1_1-complete.

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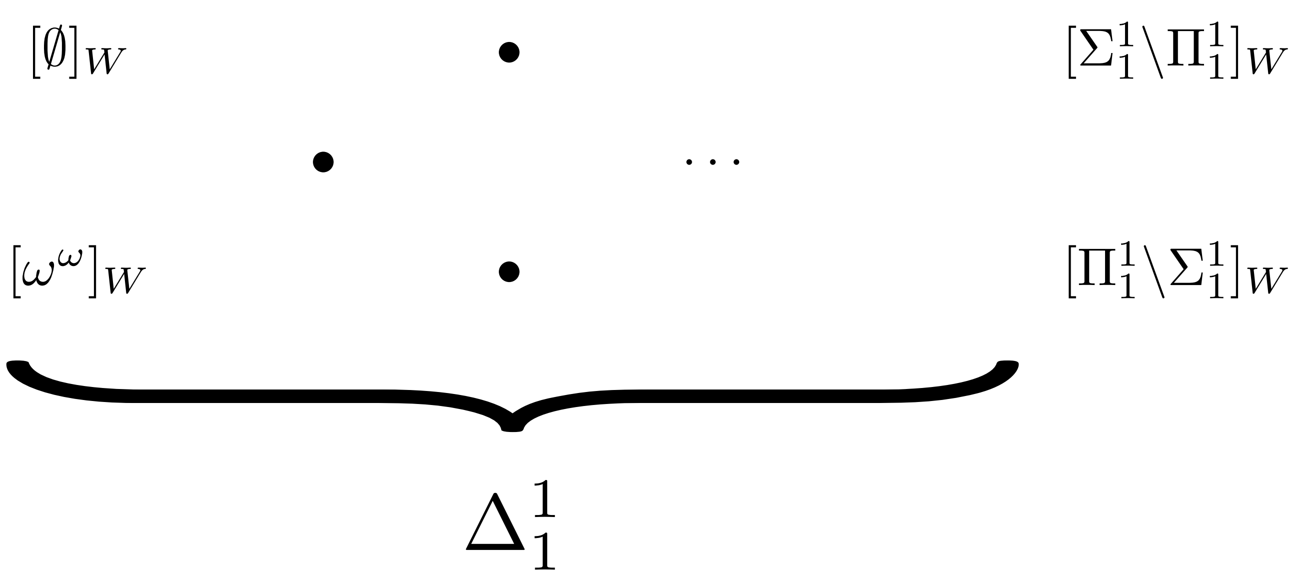

Difference Hierarchy

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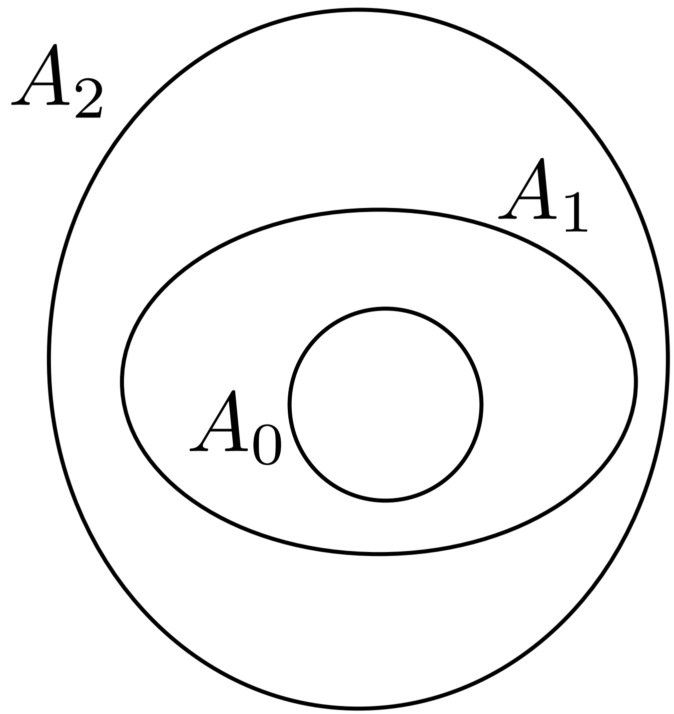

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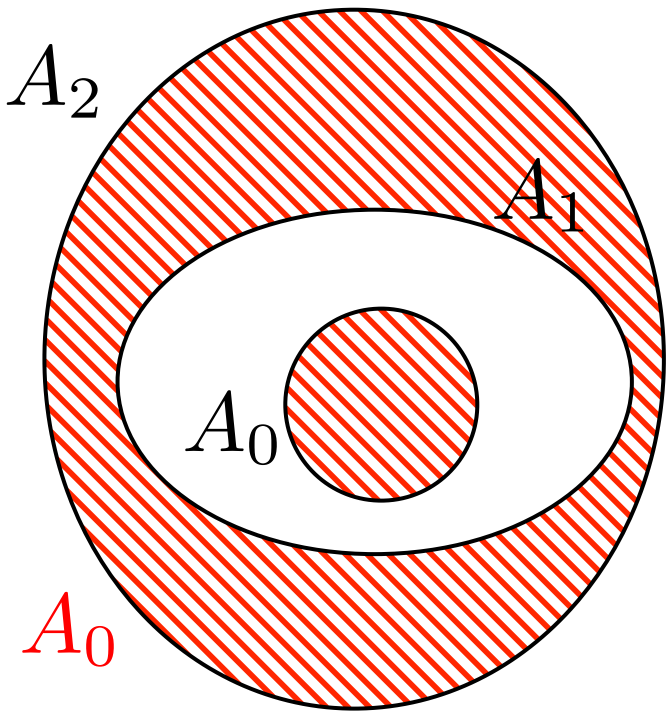
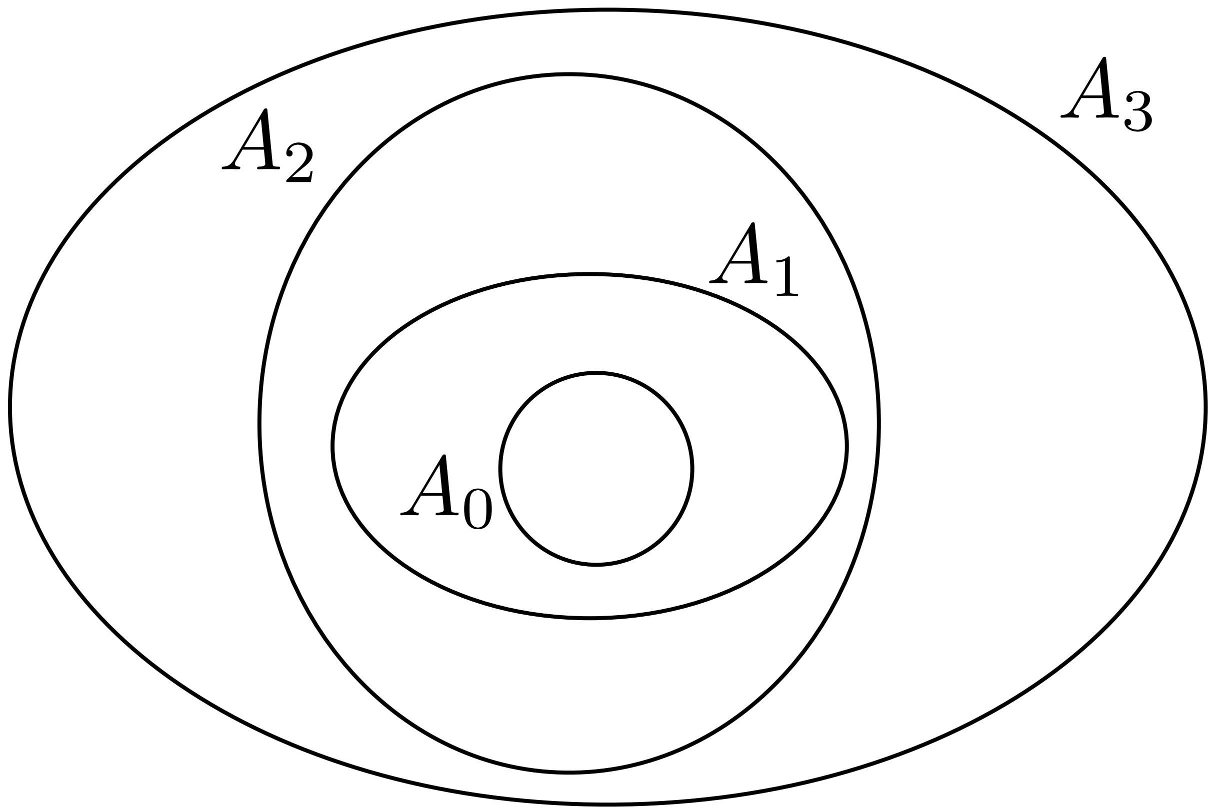


$$D_3((A_i)_{i<3}) = (A_2 \setminus A_1) \cup A_0$$

Difference Hierarchy



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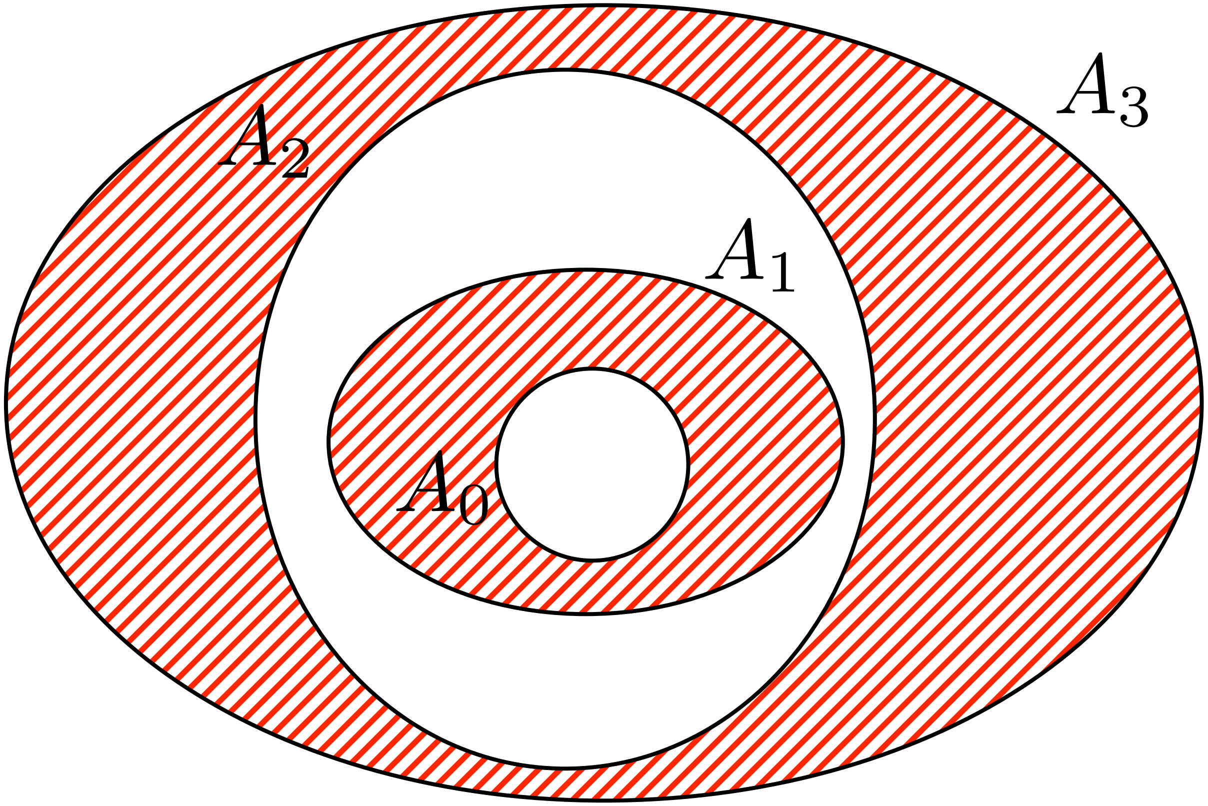

$$D_4((A_i)_{i<4}) = (A_3 \setminus A_2) \cup (A_1 \setminus A_0)$$

Difference Hierarchy

We define the classes $D_\xi(\Sigma^1_1)$ for ξ countable by:

$$D_\xi(\Sigma^1_1) = \{D_\xi((A_\eta)_{\eta<\xi}) \mid A_\eta \in \Sigma^1_1, \eta < \xi\}$$

Where:

$$D_\xi((A_\eta)_{\eta<\xi}) = \left\{x \in \bigcup_{\eta<\xi} A_\eta \quad \middle| \quad \begin{array}{l}\text{the least } \eta < \xi \text{ with } x \in A_\eta \\ \text{has parity opposite to that of } \xi\end{array}\right\}.$$

for any <u>increasing</u> sequence $(A_\eta)_{\eta<\xi}$.

We set:

$$\mathrm{Diff}(\Sigma^1_1) = \bigcup_{\xi<\omega_1} D_\xi(\Sigma^1_1)$$

Difference Hierarchy

Fact:

Assuming $\mathrm{Diff}(\Sigma^1_1)$-determinacy, if $A \in D_\xi(\Sigma^1_1) \setminus \check{D}_\xi(\Sigma^1_1)$ then A is $D_\xi(\Sigma^1_1)$-complete, for any $\xi < \omega_1$.

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$[\Sigma^1_1 \backslash \Pi^1_1]_W$ $\quad$ $[D_2(\Sigma^1_1) \backslash \check{D}_2(\Sigma^1_1)]_W$ $\quad$ $[D_3(\Sigma^1_1) \backslash \check{D}_3(\Sigma^1_1)]_W$

$\cdots$ $\quad$ $\cdots$ $\quad$ $\cdots$ $\quad$ $\cdots$

$[\Pi^1_1 \backslash \Sigma^1_1]_W$ $\quad$ $[\check{D}_2(\Sigma^1_1) \backslash D_2(\Sigma^1_1)]_W$ $\quad$ $[\check{D}_2(\Sigma^1_1) \backslash D_2(\Sigma^1_1)]_W$

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$\underbrace{\qquad\qquad}_{?}$ $\qquad$ $\underbrace{\qquad\qquad}_{?}$

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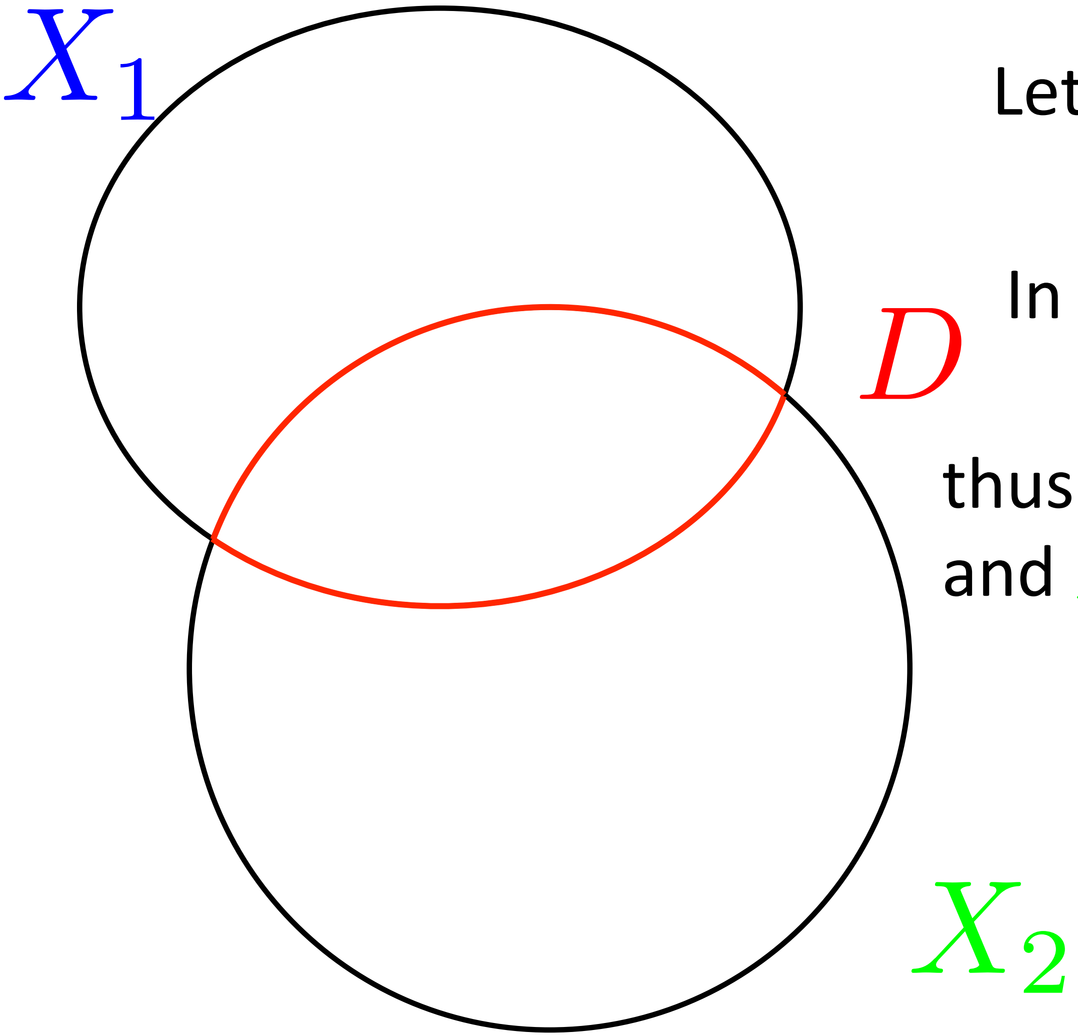
$\underbrace{\qquad\qquad}_{\Delta(D_2(\Sigma^1_1))}$ $\quad$ $\underbrace{\qquad\qquad}_{\Delta(D_3(\Sigma^1_1))}$

A description of $\Delta(D_2(\Sigma^1_1))$.



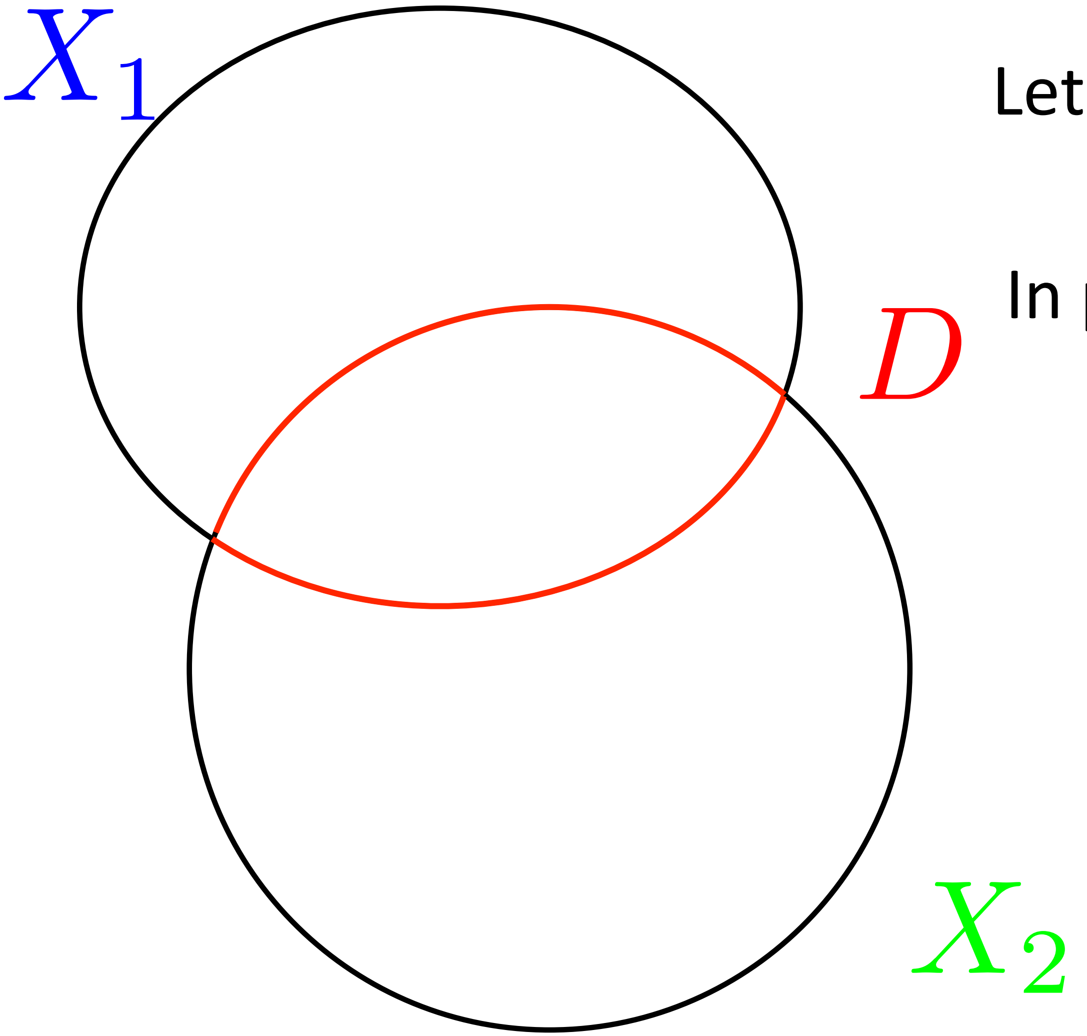
Let $D \in \Delta(D_2(\Sigma^1_1))$

In particular: $D \in D_2(\Sigma^1_1)$,

thus there exists $X_1 \in \Sigma^1_1$, and $X_2 \in \Pi^1_1$ such that:

$$D = X_2 \cap X_1$$

A description of $\Delta(D_2(\Sigma^1_1))$.



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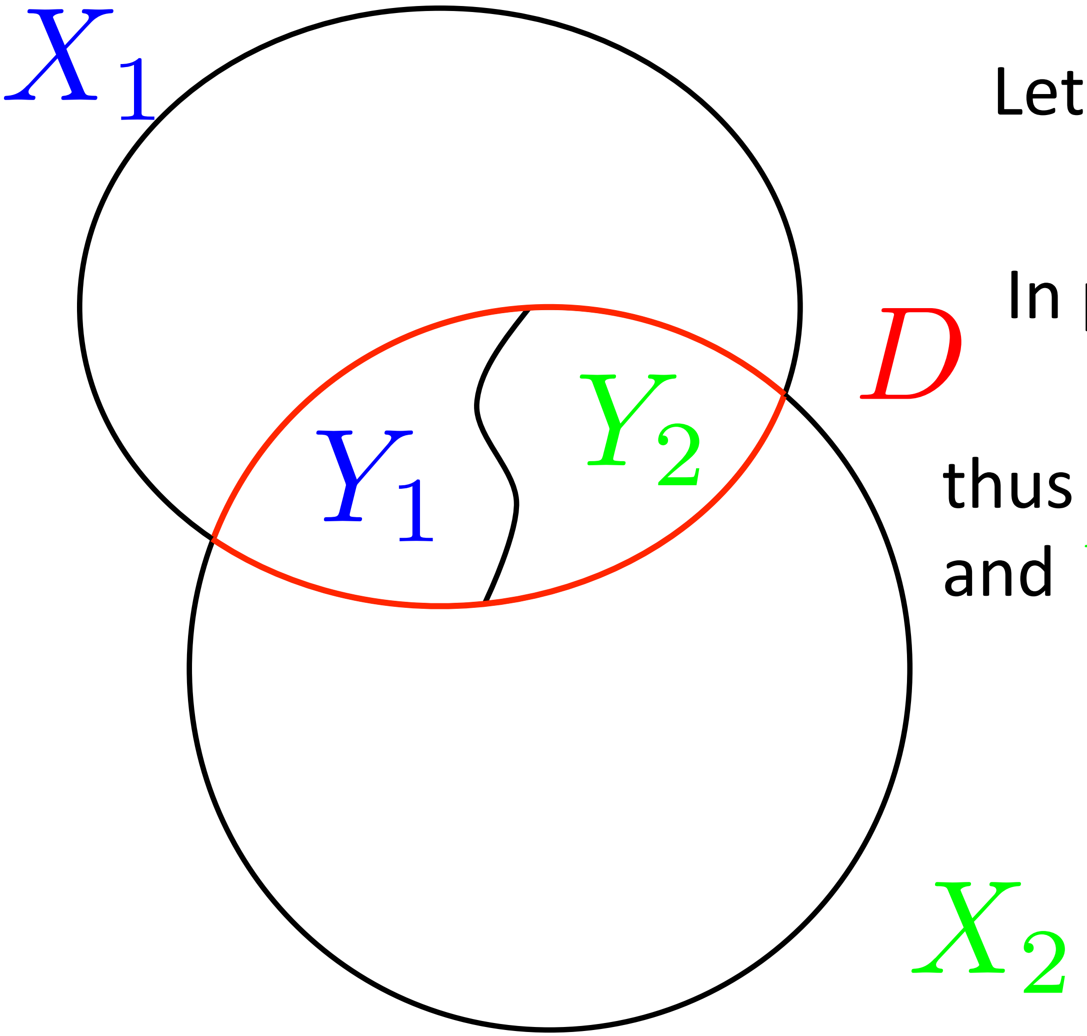


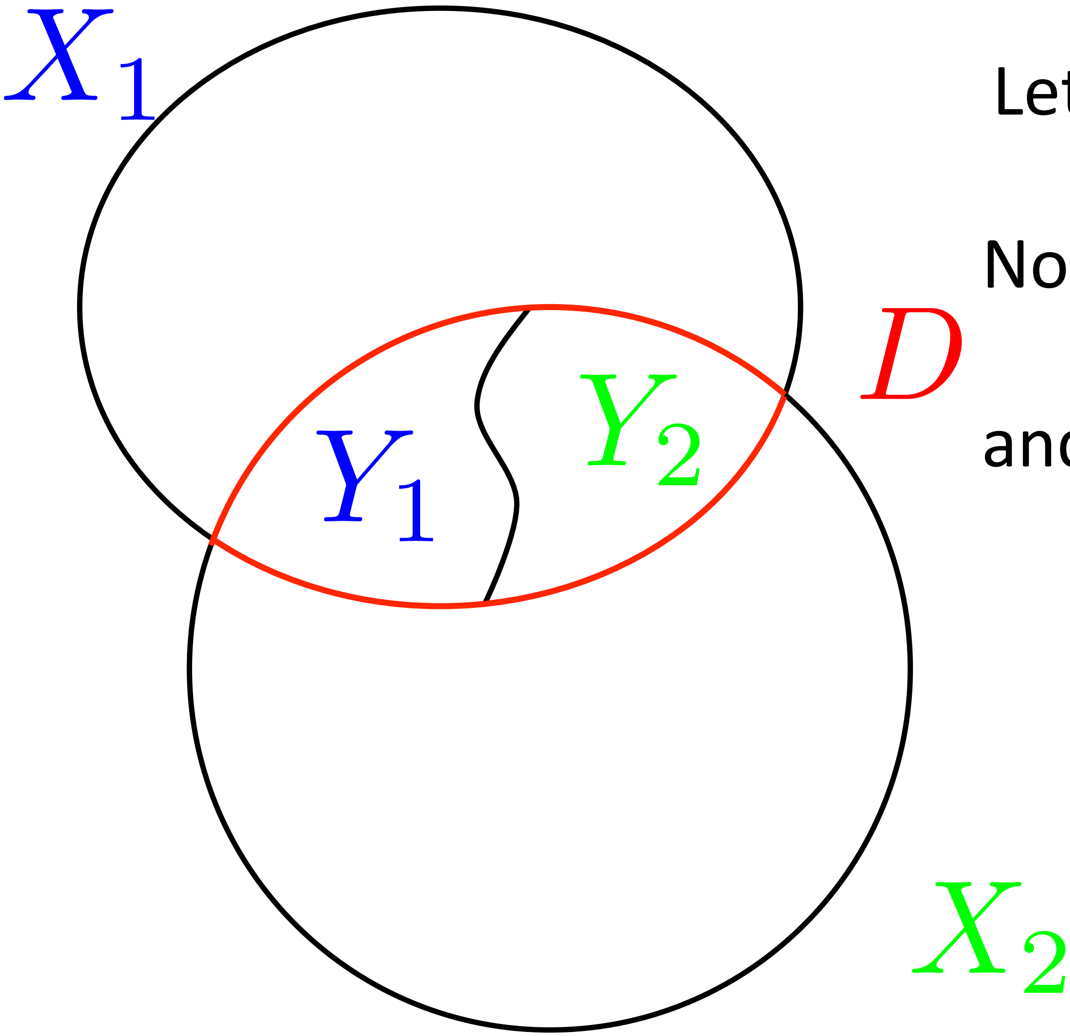
Let $D \in \Delta(D_2(\Sigma^1_1))$

In particular: $D \in \check{D}_2(\Sigma^1_1)$,

thus there exists $Y_1 \in \Sigma^1_1$, and $Y_2 \in \Pi^1_1$ such that:

$$D = Y_1 \cup Y_2$$

A description of $\Delta(D_2(\Sigma^1_1))$.



Let $D \in \Delta(D_2(\Sigma^1_1))$

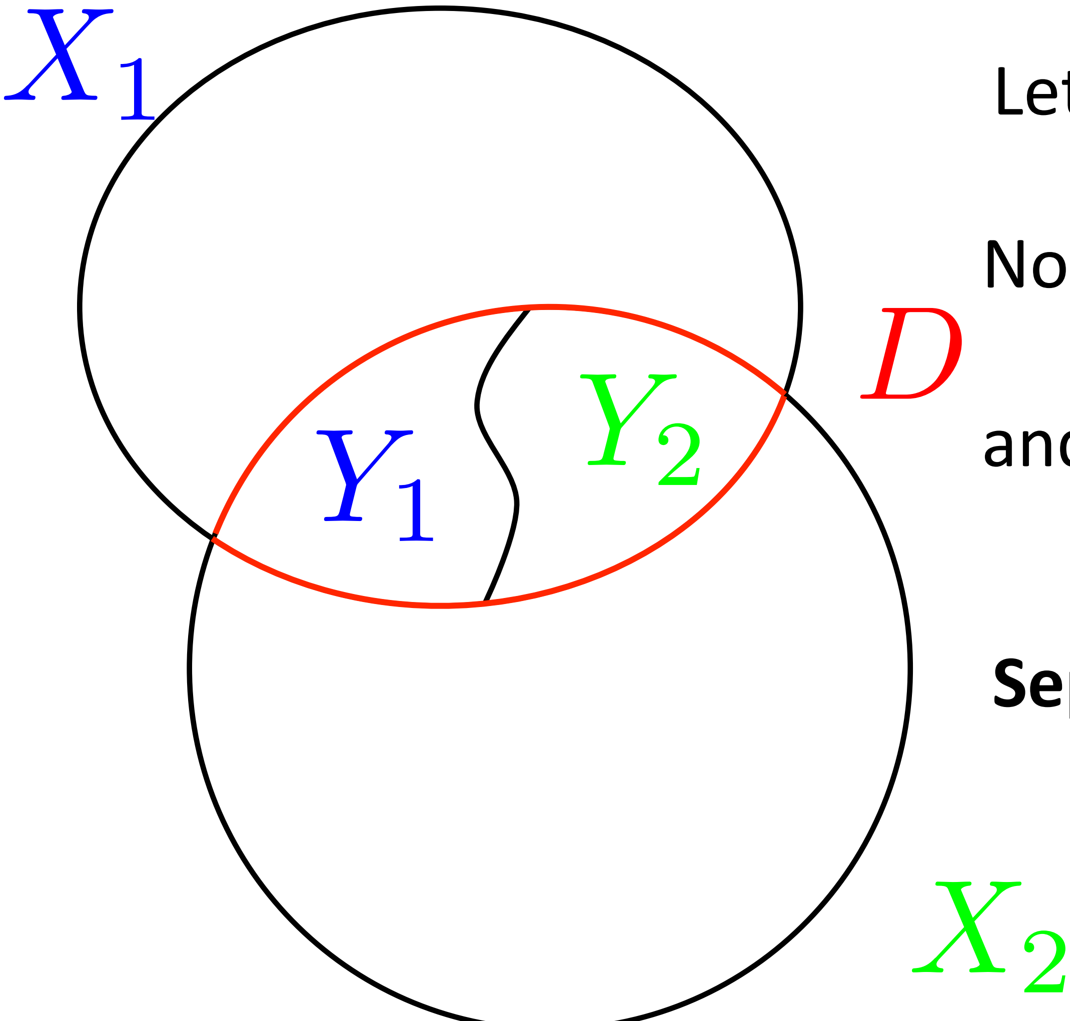
Notice that:

$$X_1 \setminus X_2 \in \Sigma^1_1,$$

and that:

$$X_1 \setminus X_2 \cap Y_1 = \emptyset.$$

A description of $\Delta(D_2(\Sigma^1_1))$.



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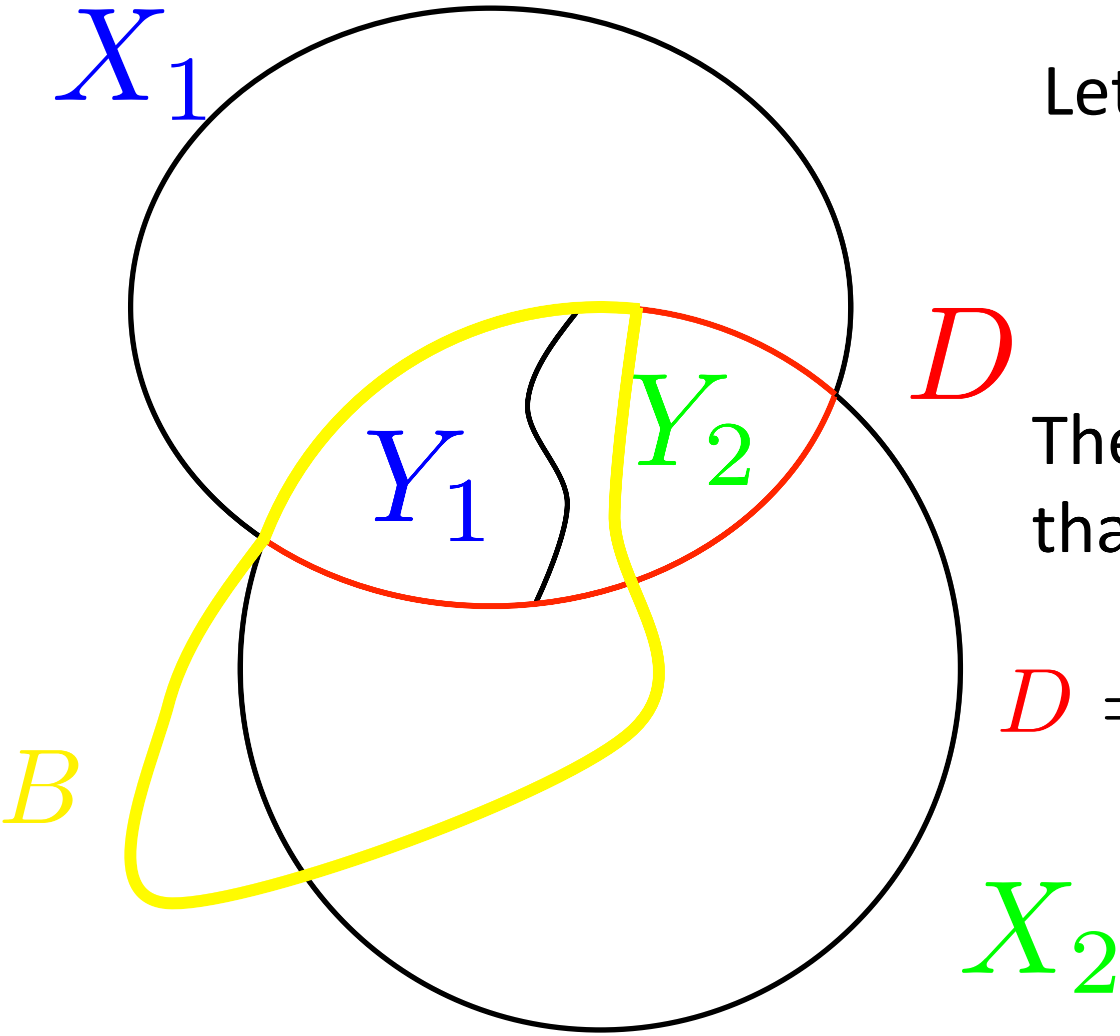
Separation Property!

A description of $\Delta(D_2(\Sigma^1_1))$.



Let $D \in \Delta(D_2(\Sigma^1_1))$

There exists B in Δ^1_1 such that:

$D = (X_1 \cap B) \cup Y_2 \setminus B$

Generalization for $\Delta(D_{\alpha+1}(\Sigma^1_1))$.

The same trick works since we have:

- If $\alpha + 1$ is even:
  - Any $D \in D_{\alpha+1}(\Sigma^1_1)$ is of the form $X_1 \cap X_2$, with $X_1 \in \Pi^1_1$ and $X_2 \in D_\alpha(\Sigma^1_1)$.
  - Any $D \in \check{D}_{\alpha+1}(\Sigma^1_1)$ is of the form $Y_1 \cup Y_2$, with $Y_1 \in \Sigma^1_1$ and $Y_2 \in \check{D}_\alpha(\Sigma^1_1)$.

Generalization for $\Delta(D_{\alpha+1}(\Sigma^1_1))$.

The same trick works since we have:

- If $\alpha + 1$ is odd:
  - Any $D \in D_{\alpha+1}(\Sigma^1_1)$ is of the form $Y_1 \cup Y_2$, with $Y_1 \in \Sigma^1_1$ and $Y_2 \in D_\alpha(\Sigma^1_1)$.
  - Any $D \in \check{D}_{\alpha+1}(\Sigma^1_1)$ is of the form $X_1 \cap X_2$, with $X_1 \in \Pi^1_1$ and $X_2 \in \check{D}_\alpha(\Sigma^1_1)$.

The limit case.

Let $D \in \Delta(D_\gamma(\Sigma^1_1))$, with γ limit. Then there exists a Borel partition $(C_i)_{i\in\omega}$ of the Baire space such that:

$$D \cap C_i \in D_{\alpha_i}(\Sigma^1_1)$$

with $\alpha_i < \gamma$.

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with $\alpha_i < \gamma$.

(To prove this result we use the **generalized reduction property** for co-analytic sets.)

Wadge Hierarchy of $\mathrm{Diff}(\Sigma^1_1)$

This analysis gives us the complete description *à la Louveau* of the Wadge hierarchy of $\mathrm{Diff}(\Sigma^1_1)$ modulo a determinacy hypothesis.

It also allows us to extend the construction of Duparc to the $\mathrm{Diff}(\Sigma^1_1)$ class.